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Entanglement and the Berry phase in a new Yang–Baxter system

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Abstract
We present a new S-matrix, a solution of the braid relation and obtain a unitary solution $\check{R}(\theta, \phi)$ of the Yang–Baxter equation (YBE) via Yang-Baxterization acting on the solution. We show that arbitrary two-qubit entangled states can be achieved by relating the unitary matrix $\check{R}(\theta, \phi)$ to entanglement. An oscillator Hamiltonian can be constructed from the $\check{R}(\theta, \phi)$ matrix. The Berry phase of the Yang–Baxter system is investigated.

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1. Introduction

Entanglement is a bizarre feature of quantum theory, and has been recognized as an important resource for applications in quantum information and computation processing [1–4]. Leveraging the entanglement and using quantum coherence, certain problems may be solved faster by a quantum computer than a classical one. The Berry phase plays an interesting role in quantum computation. Quantum-gate operations can be implemented through the geometric effects on the wavefunction of the systems; this is the so-called geometric quantum computation [5]. Geometric phases depend only on global geometric features, not on the details of evolution, such as the driving Hamiltonian, the initial and final states of the evolution [6]. Thus the geometric quantum computation is insensitive to local inaccuracies and fluctuations, which are the main resources of decoherence. Investigations on the geometric phase may produce a novel framework for quantum information science. It is shown that the geometric phase can be exploited as a tool to detect regions of criticality without having to undergo a

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quantum phase transition [7, 8]. In [9], Dajka *et al* found that the geometric phase can detect possible anisotropy of dephasing.

Recently, braiding operators and the Yang–Baxter equation (YBE) [10–12] have been introduced to the field of quantum information and quantum computation [13, 15–22]. Kitaev [13] investigated topological quantum computation (TQC) by anyons based on quantum braids. TQC [15] is one of the important approaches to achieve a fault-tolerant quantum computer. The computation scheme utilizes the topological states of non-Abelian anyons, or the particles obey non-Abelian braiding statistics. Quantum information is stored in the states with multiple quasiparticles with topological degeneracy. Quantum-gate operations are implemented by braiding the particles. Decoherence can be mitigated since unitary transformations associated with braiding particles are insensitive to the dynamics of the particles. On the other hand, Kauffman and Lomonaco [16] have explored the role of unitary braiding operators in quantum computation. It is shown that the braid matrix can be identified as the universal quantum gate [16–18]. This motivates a novel way to study quantum entanglement and the Berry phase based on the theory of braiding operators, as well as YBE. The first step along this direction is initiated by Zhang *et al* [18]. In [18], the Bell matrix generating two-qubit entangled states has been recognized to be a unitary braid transformation. Later on, an approach to describe Greenberger–Horne–Zeilinger (GHZ) states or N-qubit entangled states based on the theory of unitary braid representations has been presented in [19]. Chen and his co-workers [20, 21] used unitary braiding operators to realize entanglement swapping and generate the GHZ states, as well as the linear cluster states. With the unitary $\breve{R}(\theta,\phi)$ matrix, the authors constructed a Hamiltonian, and explored the Berry phase and quantum criticality of the Yang–Baxter system. In a very recent work [22], it has been found that any pure two-qubit entangled state can be achieved by a universal Yang–Baxter matrix.

In this paper, we present an S-matrix which is a solution of the braid relation. The S-matrix is found to be locally equivalent to the double control NOT (DCNOT) gate. By using Yang-Baxterization, we derive a unitary matrix $\breve{R}(\theta,\phi)$. Then we show that arbitrary two-qubit entangled states can be generated by the unitary matrix $\breve{R}(\theta,\phi)$. In section 3, we construct a Hamiltonian from the unitary matrix $\breve{R}(\theta,\phi)$. The Berry phase of the system is investigated. The Hamiltonian system is shown to be equivalent to an oscillator system of two fermions with frequency $\omega\cos\theta$. We end with a summary.

2. An S-matrix, Yang-Baxterization and entanglement

We first briefly review the theory of braid groups, the YBE and Yang-Baxterization approach. Let B_n denotes the braid group on n strands. B_n is generated by elementary braids $\{b_1, b_2, \ldots, b_{n-1}\}$ with the braid relations,

$$\begin{cases} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & 1 \leqslant i < n-2 \\ b_i b_j = b_j b_i & |i-j| \geqslant 2, \end{cases} \tag{1}$$

where the notation $b_i \equiv b_{i,i+1}$ is used, $b_{i,i+1}$ represents $1_1 \otimes 1_2 \otimes 1_3 \cdots \otimes S_{i,i+1} \otimes \cdots \otimes 1_n$, and 1_j is the unit matrix of the jth particle. The elementary braid b_i represents the ith string crossing over the (i+1)th string, and its inverse b_i^{-1} represents the (i+1)th string crossing over the ith string. The product of two braids $b_i b_j$ is accomplished by adjoining the top strand of b_i to the bottom strand of b_j.

As is known, a unitary solution of YBE can be found via Yang-Baxterization acting on the solution of the braid relation. For example, if b_i has two eigenvalues, then the Yang-

Baxterization of the unitary braiding operator b_i is

$$\check{R}_i(x) = \frac{1}{\sqrt{1+x^2}}\left(b_i + x b_i^{-1}\right), \tag{2}$$

where $\check{R}_i$ is short for $\check{R}_{i,i+1}$. The unitary $\check{R}$-matrix satisfies the YBE which is of the form

$$\check{R}_i(x)\check{R}_{i+1}(xy)\check{R}_i(y) = \check{R}_{i+1}(y)\check{R}_i(xy)\check{R}_{i+1}(x), \tag{3}$$

where multiplicative parameters x and y are known as the spectral parameters. The asymptotic behavior of $\check{R}(x)$ is x-independent, that is $\lim_{x\to\infty}\check{R}_i(x) = b_i^{-1}$. Generally, multi-spin interaction Hamiltonians can be constructed based on the YBE. As $\check{R}$ is unitary, it can define the evolution of a state $|\Psi(0)\rangle$:

$$|\Psi(t)\rangle = \check{R}_i(t)|\Psi(0)\rangle, \tag{4}$$

where $\check{R}_i(t)$ is time dependent, which can be realized by specifying a corresponding time-dependent parameter of $\check{R}_i$. By taking partial derivative of the state $|\Psi(t)\rangle$ with respect to time t, we have an equation,

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} &= \mathrm{i}\hbar\left[\frac{\partial\check{R}_i(t)}{\partial t}\check{R}_i^{\dagger}(t)\right]\check{R}_i(t)|\Psi(0)\rangle \\ &= H(t)|\Psi(t)\rangle, \end{aligned} \tag{5}$$

where $H(t) = \mathrm{i}\hbar\frac{\partial|\check{R}_i(t)\rangle}{\partial t}\check{R}_i^{\dagger}(t)$ is the Hamiltonian governing the evolution of the state $|\Psi(t)\rangle$. Thus, the Hamiltonian $H(t)$ for the Yang–Baxter system is derived through the Yang-Baxterization approach.

In the following, we present a solution of the braid relation. Generally, the standard eight-vertex model is a generalization of the 'ice model'. In this model, each vertex can be represented by a matrix element which is explained as the Boltzmann weight. In [18], the authors gave up the nonnegativity condition for the Boltzmann weight, and obtained some useful quantum gates which satisfy the YBE. Motivated by this, we give up the nonnegative condition and alter the location of the matrix elements of the model. It is hoped that this may give some interesting results. The S-matrix takes the following form:

$$S = \begin{pmatrix} 0 & a_1 & a_2 & 0 \\ a_3 & 0 & 0 & a_4 \\ a_5 & 0 & 0 & a_6 \\ 0 & a_7 & a_8 & 0 \end{pmatrix}, \tag{6}$$

where a_i $(i = 1, \ldots, 8)$ are the parameters to be determined. Setting $a_1a_3 = a_2a_5 = a_4a_7 = a_6a_8 = \frac{1}{2}$, we have $a_1 = a_4$ and $a_2 = a_6$. From the relation $S^2 = 1$, it is obtained that $a_1^2 = -a_2^2$. In the case of $a_1 = -\mathrm{i}a_2 = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\phi}$, a new S-matrix is found to be of the form

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\phi} & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi} & 0 \\ \mathrm{e}^{-\mathrm{i}\phi} & 0 & 0 & \mathrm{e}^{\mathrm{i}\phi} \\ -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi} & 0 & 0 & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi} \\ 0 & \mathrm{e}^{-\mathrm{i}\phi} & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\phi} & 0 \end{pmatrix}, \tag{7}$$

where the parameter ϕ is real. One can verify that $S^2 = I$ and $S^{\dagger}S = SS^{\dagger} = I$, thus the S-matrix is unitary. For ith and $(i+1)$th lattices, S can be expressed in terms of spin operators,

$$\begin{aligned} S = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\phi}&\left[\frac{1+\mathrm{i}}{2}\left(S_i^+ + S_{i+1}^+\right) + (1-\mathrm{i})\left(S_i^3S_{i+1}^+ - S_i^+S_{i+1}^3\right)\right] \\ &+ \frac{1}{\sqrt{2}}\,\mathrm{e}^{-\mathrm{i}\phi}\left[\frac{1-\mathrm{i}}{2}\left(S_i^- + S_{i+1}^-\right) + (1+\mathrm{i})\left(S_i^3S_{i+1}^- - S_i^-S_{i+1}^3\right)\right], \end{aligned} \tag{8}$$

where $S_i^+ = S_i^1 + \mathrm{i}S_i^2$ and $S_i^- = S_i^1 - \mathrm{i}S_i^2$ are the raising and lowering operators of spin-1/2 angular momentum for the ith particle, respectively. The braid relation (1) and $S^2 = I$ are similar to those for the usual permutation operator $P_{i,i+1} = \frac{1}{2}(1 + \vec{\sigma}_i \cdot \vec{\sigma}_{i+1})$, where $\vec{\sigma}$ denotes Pauli matrices. Since the permutation operators P and S do not have the same eigenvalues, one cannot transfer from one to another by unitary transformations. So one can say that S is a new braiding matrix. Unitary braid matrix can be construed as a quantum gate [16]. The S-matrix is calculated to be locally equivalent to the DCNOT gate in the following way:

$$\mathrm{DCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (A \otimes B) \cdot S \cdot (C \otimes D), \tag{9}$$

where

$$\begin{aligned} A &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} \\ \mathrm{i} & \mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} \end{pmatrix}, \qquad & B &= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} & \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} \\ 1 & -\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}} \end{pmatrix}, \\ C &= \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} & \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \\ 1 & 1 \end{pmatrix}, \qquad & D &= \begin{pmatrix} -1 & 0 \\ 0 & -\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}} \end{pmatrix}. \end{aligned} \tag{10}$$

We next derive a unitary matrix $\check{R}$ from S by the Yang-Baxterization approach. As follows, we write the YBE in the form of additive spectral parameters u and ν :

$$\check{R}_i(u)\check{R}_{i+1}(u+\nu)\check{R}_i(\nu) = \check{R}_{i+1}(\nu)\check{R}_i(u+\nu)\check{R}_{i+1}(u). \tag{11}$$

The asymptotic behavior of $\check{R}(u)$ is u-independent, that is $\lim_{u\to\infty}\check{R}_i(u) = b_i$. From a given solution of the braid relation S, a unitary matrix $\check{R}(u)$ can be constructed by using the approach of Yang-Baxterization. It is easy to show that $\check{R}(u) = \rho(u)(I + \mathrm{i}uS)$ is a rational solution of YBE (u is real), where $\rho(u)$ is a normalization factor. One can choose appropriate $\rho(u)$ to ensure that $\check{R}(u)$ is unitary. According to the inverse scattering method, $\check{R}^{-1}(u)$ is proportional to $\check{R}(-u)$. For the purpose of finding a unitary matrix $\check{R}(u)$, $\check{R}^\dagger(u)$ should be equal to the inverse matrix of $\check{R}(u)$ or $\check{R}^{-1}(u)$. As a result, we obtain the unitary $\check{R}(u)$ matrix written in terms of the S-matrix, $\check{R}(u) = \frac{1}{\sqrt{1+u^2}}(I + \mathrm{i}uS)$. By introducing a new variable θ with $\cos\theta = \frac{u}{\sqrt{1+u^2}}$ and $\sin\theta = \frac{1}{\sqrt{1+u^2}}$, the matrix $\check{R}(u)$ can be rewritten as

$$\check{R}(\theta,\phi) = \sin\theta I + \mathrm{i}\cos\theta S. \tag{12}$$

There are two parameters θ and ϕ in the unitary $\check{R}(\theta,\phi)$ matrix. We now show that an arbitrary two-qubit entangled state is achievable based on the unitary matrix $\check{R}(\theta,\phi)$. When $\check{R}(\theta,\phi)$ acts on the direct product states $|kl\rangle \equiv |k\rangle_i \otimes |l\rangle_{i+1} (k,l = 0,1)$, the $\check{R}(\theta,\phi)$ matrix transfers the two-qubit product states to entangled states,

$$\begin{aligned} |00\rangle &\to \sin\theta|00\rangle + \frac{\mathrm{i}}{\sqrt{2}}\cos\theta\,\mathrm{e}^{-\mathrm{i}\phi}|01\rangle + \frac{1}{\sqrt{2}}\cos\theta\,\mathrm{e}^{-\mathrm{i}\phi}|10\rangle, \\ |01\rangle &\to \frac{\mathrm{i}}{\sqrt{2}}\cos\theta\,\mathrm{e}^{\mathrm{i}\phi}|00\rangle + \sin\theta|01\rangle + \frac{\mathrm{i}}{\sqrt{2}}\cos\theta\,\mathrm{e}^{-\mathrm{i}\phi}|11\rangle, \\ |10\rangle &\to -\frac{1}{\sqrt{2}}\cos\theta\,\mathrm{e}^{\mathrm{i}\phi}|00\rangle + \sin\theta|10\rangle + \frac{1}{\sqrt{2}}\cos\theta\,\mathrm{e}^{-\mathrm{i}\phi}|11\rangle, \\ |11\rangle &\to \frac{\mathrm{i}}{\sqrt{2}}\cos\theta\,\mathrm{e}^{\mathrm{i}\phi}|01\rangle - \frac{1}{\sqrt{2}}\cos\theta\,\mathrm{e}^{\mathrm{i}\phi}|10\rangle + \sin\theta|11\rangle. \end{aligned} \tag{13}$$

Let us find the entanglement degree of the above states by using concurrence [23]. The concurrence is defined as

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

where λ_i are the eigenvalues of $\mathrm{R} = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ taken in decreasing order, $\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$ with ρ^* being the complex conjugate of ρ, and σ_y is the Pauli spin matrix. For a pure two-qubit state, $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, the concurrence can be found to be

$$C(\psi) = 2|ad - bc|. \tag{14}$$

It is not difficult to obtain the entanglement degree of the four entangled states (13) as follows:

$$C(1) = C(2) = C(3) = C(4) = \cos^2\theta, \tag{15}$$

where $C(i)(i = 1, 2, 3, 4)$ denote the ith state's concurrence. It is worth noting that the four states in the right-hand side of equation (13) possess the same entanglement degree which depends on the parameter θ. When u increases from minus infinity to plus infinity, θ varies from π to 0 correspondingly. As u goes to infinity, the $\check{R}(\theta, \phi)$ matrix reduces to iS, and the four states' entanglement reaches the maximum value of 1. If θ takes other values, the states possess continuous entanglement degree determined by θ. So one can say that the unitary S-matrix describes the maximum entangled states, and the Yang–Baxter matrix $\check{R}(\theta, \phi)$ generates entangle states with arbitrary degree of entanglement. When $\theta = 0/\pi$, maximally entangled states can be achieved. It is possible to find an explicit form of the solution of the YBE to generate the maximally entangled states. By choosing $\theta = 0$ and $\phi = 0$, we have

$$\check{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathrm{i} & -1 & 0 \\ \mathrm{i} & 0 & 0 & \mathrm{i} \\ 1 & 0 & 0 & -1 \\ 0 & \mathrm{i} & 1 & 0 \end{pmatrix}. \tag{16}$$

The action of the $\check{R}$-matrix results in the two-qubit maximally entangled states.

It is worth mentioning that the concurrences $C(i)$ do not depend on the parameter ϕ. This tells us that the effect of ϕ can be reduced by local unitary operation U. That is

$$(U \otimes U)S(U^{-1} \otimes U^{-1}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & \mathrm{i} & 0 \\ 1 & 0 & 0 & 1 \\ -\mathrm{i} & 0 & 0 & \mathrm{i} \\ 0 & 1 & -\mathrm{i} & 0 \end{pmatrix}, \tag{17}$$

where $U = \begin{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}\phi}{2}} & 0 \\ 0 & \mathrm{e}^{\frac{\mathrm{i}\phi}{2}} \end{pmatrix}$, and hence we have ϕ-independent $\check{R}(\theta)$ as follows:

$$\check{R}(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\sin\theta & \mathrm{i}\cos\theta & -\cos\theta & 0 \\ \mathrm{i}\cos\theta & \sqrt{2}\sin\theta & 0 & \mathrm{i}\cos\theta \\ \cos\theta & 0 & \sqrt{2}\sin\theta & -\cos\theta \\ 0 & \mathrm{i}\cos\theta & \cos\theta & \sqrt{2}\sin\theta \end{pmatrix}. \tag{18}$$

3. Hamiltonian and Berry phase

A Hamiltonian of the Yang–Baxter system can be constructed from the $\check{R}(\theta, \phi)$ matrix. As shown in [20], the Hamiltonian is obtained through the Schrödinger evolution of the entangled states. Let the parameter ϕ be time dependent as $\phi = \omega t$ and θ be time independent, the Hamiltonian is

$$\begin{aligned} \hat{H} &= \mathrm{i}\hbar \frac{\mathrm{d}\check{R}(\theta, \phi)}{\mathrm{d}t} \check{R}^\dagger(\theta, \phi) \\ &= \mathrm{i}\hbar \cos\theta \frac{\partial S}{\partial t}(\mathrm{i}\sin\theta I + \cos\theta S). \end{aligned} \tag{19}$$

In terms of the standard basis of $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$, the Hamiltonian is of the following form:

$$\hat{H} = \hbar\dot{\phi}\cos\theta \begin{pmatrix} -\cos\theta & -\frac{\mathrm{i}}{\sqrt{2}}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & \frac{1}{\sqrt{2}}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & 0 \\ \frac{\mathrm{i}}{\sqrt{2}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} & 0 & \mathrm{i}\cos\theta & -\frac{\mathrm{i}}{\sqrt{2}}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} \\ \frac{1}{\sqrt{2}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} & -\mathrm{i}\cos\theta & 0 & \frac{1}{\sqrt{2}}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} \\ 0 & \frac{\mathrm{i}}{\sqrt{2}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} & \frac{1}{\sqrt{2}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} & \cos\theta \end{pmatrix}. \tag{20}$$

The eigenstates of the Yang–Baxter system are found to be

$$\begin{aligned} |\psi_1\rangle &= \sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}|00\rangle + \mathrm{i}\frac{\sqrt{2}}{2}\cos\frac{\theta}{2}|01\rangle + \frac{\sqrt{2}}{2}\cos\frac{\theta}{2}|10\rangle \\ |\psi_2\rangle &= -\mathrm{i}\frac{\sqrt{2}}{2}\sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}|01\rangle + \frac{\sqrt{2}}{2}\sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}|10\rangle + \cos\frac{\theta}{2}|11\rangle \\ |\psi_3\rangle &= -\cos\frac{\theta}{2}|00\rangle + \mathrm{i}\frac{\sqrt{2}}{2}\sin\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\phi}|01\rangle + \frac{\sqrt{2}}{2}\sin\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\phi}|10\rangle \\ |\psi_4\rangle &= \mathrm{i}\frac{\sqrt{2}}{2}\cos\frac{\theta}{2}|01\rangle - \frac{\sqrt{2}}{2}\cos\frac{\theta}{2}|10\rangle + \sin\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\phi}|11\rangle \end{aligned} \tag{21}$$

with the corresponding eigenvalues $E_1 = E_2 = \hbar\omega\cos\theta$, $E_3 = E_4 = -\hbar\omega\cos\theta$. According to the definition of the Berry phase [6], when the parameter ϕ evolves adiabatically from 0 to 2π, the Berry phase accumulated by the states $|\psi_i\rangle (i = 1, 2, 3, 4)$ is

$$\gamma_i = \mathrm{i}\int_0^{2\pi} \langle\psi_i|\frac{\mathrm{d}}{\mathrm{d}\phi}|\psi_i\rangle\,\mathrm{d}\phi. \tag{22}$$

Using equations (21) and (22), we obtain the Berry phases for the entangled states $|\psi_i\rangle$:

$$\begin{cases} \gamma_1 = \gamma_2 = -\pi(1-\cos\theta) = -\dfrac{\Omega}{2} \\ \gamma_3 = \gamma_4 = \pi(1-\cos\theta) = \dfrac{\Omega}{2}, \end{cases} \tag{23}$$

where $\Omega = 2\pi(1-\cos\theta)$ is the solid angle enclosed by the loop on the Bloch sphere.

Introducing three operators

$$\begin{aligned} S^+ &= \frac{1}{\sqrt{2}}\left[\frac{\mathrm{i}-1}{2}\left(S_i^+ + S_{i+1}^+\right) + (1+\mathrm{i})\left(S_i^3 S_{i+1}^+ - S_i^+ S_{i+1}^3\right)\right], \\ S^- &= \frac{1}{\sqrt{2}}\left[\frac{-1-\mathrm{i}}{2}\left(S_i^- + S_{i+1}^-\right) + (1-\mathrm{i})\left(S_i^3 S_{i+1}^- - S_i^- S_{i+1}^3\right)\right], \\ S^3 &= \frac{1}{2}\left[\left(S_i^3 + S_{i+1}^3\right) - \mathrm{i}\left(S_i^+ S_{i+1}^- - S_i^- S_{i+1}^+\right)\right], \end{aligned} \tag{24}$$

it is not difficult to find that $(S^{\pm})^2 = 0$ and $(S^3)^2 = \frac{1}{4}$. We have a $SU(2)$ group formed by the three operators, fulfilling conditions $[S^+, S^-] = 2S^3$ and $[S^3, S^{\pm}] = \pm S^{\pm}$. The Hamiltonian (20) can be rewritten based on the operators (24) as follows:

$$\begin{aligned} \hat{H} &= -\hbar\omega\cos\theta(2\cos\theta S^3 + \sin\theta\,\mathrm{e}^{\mathrm{i}\phi}S^+ + \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}S^-) \\ &= -\hbar\omega\cos\theta\hat{H}_0, \end{aligned} \tag{25}$$

where $\hat{H}_0$ is of the form

$$\hat{H}_0 = 2\cos\theta S^3 + \sin\theta\,\mathrm{e}^{\mathrm{i}\phi}S^+ + \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}S^-.$$

Thus, the Hamiltonian $\hat{H}$ constructed from the matrix $\check{R}(\theta, \phi)$, or the S-matrix has the same physical meaning as that given in [21]. That is, $\hat{H}$ is an oscillator Hamiltonian of two fermions with frequency $\omega \cos\theta$. For the Yang–Baxter system, $\theta = 0$ is a critical point. In other words, the Hamiltonian represents a standard oscillator when $\theta = 0$, and when $\theta \neq 0$ the wavefunction of the system is described by the spin-coherent state [24]. The quantum criticality can be captured by the Berry phase of the system since the Berry phase $\pm\pi(1 - \cos\theta) = 0$ at the critical point $\theta = 0$. Our result is consistence with that given in [21].

4. Summary

In this paper, we have presented a new braiding operator S and derived a unitary $\check{R}(\theta, \phi)$ matrix via Yang-Baxterization of the new S-matrix. The S-matrix is found to be locally equivalent to the DCNOT gate. We show that any pure two-qubit entangled states can be achieved when the unitary $\check{R}(\theta, \phi)$ matrix acts on the direct product states. Specifically, the braiding operator S describes the maximally entangled states, or the Bell states, while the Yang–Baxter matrix $\check{R}(\theta, \phi)$ generates entangled states with arbitrary degree of entanglement. The evolution of the Yang–Baxter system is explored by constructing a Hamiltonian from the unitary $\check{R}(\theta, \phi)$ matrix, and the Hamiltonian is shown to describe an oscillator of two fermions with frequency $\omega \cos\theta$. We study the Berry phase of the Yang–Baxter system and find that the Berry phase captures the quantum criticality of the system.

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